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Disturbed Zhang Dynamics Control for Triple-Integrator to Multiple-Integrator Systems: Design Formula Collection, Error Dynamics Equivalence, and Theoretical Analyses

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Abstract: In this paper, Zhang dynamics (ZD) is investigated and applied in a multiple-integrator system (MIS) with or without noise disturbance for tracking control. For illustration, the design procedure of ZD control (ZDC) for a triple-integrator system (TIS) is shown. Besides, the error dynamics of the TIS synthesized by the ZDC is presented, and its equivalent system comes to light. Moreover, theoretical analyses on the TIS equipped with the ZDC are given to substantiate its stability and convergence. For more generalization, the ZDC for the MIS is developed, and the corresponding error dynamics is exhibited. In addition, theoretical analyses on the MIS equipped with the ZDC are provided. Specifically, regardless of noise pollution, the tracking error and other ZD errors of the MIS synthesized by the ZDC are bounded and convergent.

Key Words: Zhang Dynamics Control, Multiple-Integrator System, Noise Disturbance, Tracking Error, Stability

1 Introduction and Preliminaries

Zhang dynamics (ZD), proposed by Zhang *et al*, is a simple and effective method to design the control input [1]. Hitherto, ZD has been applied in many fields, such as Genesio chaotic system synchronization [2], time-varying reciprocal [3], tracking control [1, 4, 5]. There is no doubt that ZD plays a significant role in tracking control, which is one of typical and important issues in control field. The multiple-integrator system (MIS) is one classic kind of linear systems, and it is widely investigated in control theory with analyses and practical applications [6–12]. In this paper, we design ZD control (ZDC) for a triple-integrator system (TIS). Moreover, we generalize the ZDC for the MIS.

According to previous work [6, 12], the nth-order MIS without noise disturbance is written as

$$\begin{cases} \dot{x}_i = x_{i+1}, \text{ with } i = 1, 2, \cdots, n-1, \\ \dot{x}_n = u, \end{cases} \tag{1}$$

where each $x_i \in \mathbb{R}$ is a state of the MIS and $u \in \mathbb{R}$ is the control input. Note that noises are inevitable in the practical implementation. They are caused by many factors, such as modeling errors, parameter errors, computational errors, external disturbance, and so on. Generally, all noises can be regarded as additive noises [13]. From this perspective, the MIS with noise disturbance is presented as

$$\begin{cases} \dot{x}_i = x_{i+1}, \text{ with } i = 1, 2, \cdots, n-1, \\ \dot{x}_n = u + g, \end{cases} \tag{2}$$

where $g \in \mathbb{R}$ is the sum of all noise disturbance, including random noises, time-varying bias errors, and constant implementation errors.

The remainder of this paper is organized as follows. In Section 2, the design procedure of the ZDC for the TIS is illustrated by a collection of all design formulas. Then, in Section 3, we discover the error dynamics equivalence of the TIS synthesized by the ZDC. Theoretical analyses on the stability and the convergence of the TIS equipped with the ZDC are given in Section 4. The design control input for the MIS and the corresponding theoretical analyses are presented in Section 5. Section 6 finally concludes the paper. The main contributions of the paper are shown as below.

1) Under the assistance of ZDC, the stability of the MIS with or without noise disturbance is converted into the stability of its error dynamics, which is composed of the state equations of the tracking error and other ZD errors.
2) Theoretical analyses prove that, by utilizing ZD method to design the control input, the tracking errors and other ZD errors of the MIS are bounded and convergent, which means that the MIS with or without noise disturbance is stable and convergent.
3) While verifying the consistency of the ZDC, we discover a system that is equivalent to the error dynamics of the TIS with the design control input.

2 Design Formula Collection of ZDC for TIS

First, we take the TIS as a simple example. On the basis of (1), the TIS without noise disturbance is written as

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = u. \end{cases} \tag{3}$$

Based on (2), the TIS with noise disturbance is formulated as follows:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = u + g. \end{cases} \tag{4}$$

For convenience, x_1 is set as the output $y = x_1$ of the TIS to track a reference trajectory γ. By adopting the ZD method [2], we design the simple, effective, and robust ZDC for the TIS. The procedure of the ZDC for the TIS is shown by the following design formula collection:

$$e_1 = y - \gamma \in \mathbb{R},$$

$$
\begin{aligned}
\dot{e}_1 &= -\lambda_1 e_1 \in \mathbb{R}, \text{ with } \lambda_1 \in \mathbb{R}^+,\\
x_2 - \dot{\gamma} &= -\lambda_1(x_1 - \gamma) \in \mathbb{R},\\
e_2 &= \dot{e}_1 + \lambda_1 e_1 \in \mathbb{R},\\
\dot{e}_2 &= -\lambda_2 e_2 \in \mathbb{R}, \text{ with } \lambda_2 \in \mathbb{R}^+,\\
x_3 - \ddot{\gamma} &= -(\lambda_1+\lambda_2)\dot{e}_1 - \lambda_1\lambda_2 e_1 \in \mathbb{R},\\
e_3 &= \dot{e}_2 + \lambda_2 e_2 \in \mathbb{R},\\
\dot{e}_3 &= -\lambda_3 e_3 \in \mathbb{R}, \text{ with } \lambda_3 \in \mathbb{R}^+,\\
\dot{x}_3 - \dddot{\gamma} &= -(\lambda_1+\lambda_2+\lambda_3)\ddot{e}_1 - (\lambda_1\lambda_2+\lambda_1\lambda_3+\\
&\quad \lambda_2\lambda_3)\dot{e}_1 - \lambda_1\lambda_2\lambda_3 e_1 \in \mathbb{R}.
\end{aligned} \tag{5}
$$

Finally, on the basis of equation group (3) and equation (5), the ZDC for the TIS is formulated as

$$
\begin{aligned}
u =& \dddot{\gamma} - (\lambda_1+\lambda_2+\lambda_3)\ddot{e}_1 - (\lambda_1\lambda_2+\lambda_1\lambda_3+\\
&\lambda_2\lambda_3)\dot{e}_1 - \lambda_1\lambda_2\lambda_3 e_1 \in \mathbb{R}.
\end{aligned} \tag{6}
$$

3 Error Dynamics Equivalence of TIS Equipped with ZDC

Through the design procedure of the ZDC (6) for the TIC, we discover that the stability of the TIS with the control input (6) has been converted into the convergence of the errors e_1, e_2, and e_3. Therefore, the error dynamics of the TIS synthesized by the ZDC (6) is composed of the state equations of e_1, e_2, and e_3.

In light of the design procedure of the control input (6) for the TIS, we obtain the state equations of e_1, e_2, and e_3 readily:

$$
\begin{cases}
\dot{e}_1 = e_2 - \lambda_1 e_1,\\
\dot{e}_2 = e_3 - \lambda_2 e_2,\\
\dot{e}_3 = -\lambda_3 e_3.
\end{cases} \tag{7}
$$

Moreover, equation group (7) can be transformed into

$$
\dot{\mathbf{e}} = \begin{pmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\end{pmatrix} = \begin{pmatrix}-\lambda_1 & 1 & 0\\ 0 & -\lambda_2 & 1\\ 0 & 0 & -\lambda_3\end{pmatrix}\begin{pmatrix}e_1\\ e_2\\ e_3\end{pmatrix} = A_3\mathbf{e}. \tag{8}
$$

Similarly, the error dynamics of the disturbed TIS (4) with the control input (6) for tracking control is expressed as

$$
\dot{\mathbf{e}} = \begin{pmatrix}-\lambda_1 & 1 & 0\\ 0 & -\lambda_2 & 1\\ 0 & 0 & -\lambda_3\end{pmatrix}\begin{pmatrix}e_1\\ e_2\\ e_3\end{pmatrix} + \begin{pmatrix}0\\ 0\\ 1\end{pmatrix} g = A_3\mathbf{e} + B_3 g. \tag{9}
$$

To substantiate that the ZDC (6) is consistent with the design procedure, we define the following equation group:

$$
\begin{cases}
z_1 = e_1,\\
z_2 = \dot{z}_1 = \dot{e}_1,\\
z_3 = \dot{z}_2 = \ddot{z}_1 = \ddot{e}_1,
\end{cases} \tag{10}
$$

which is inspired by the way in which a high-order ordinary differential equation is solved. Then, the state equations of z_1, z_2, and z_3 are presented as

$$
\begin{cases}
\dot{z}_1 = \dot{e}_1 = z_2,\\
\dot{z}_2 = \ddot{e}_1 = z_3,\\
\dot{z}_3 = \dddot{e}_1.
\end{cases} \tag{11}
$$

Based on (6), we have the following deduction for $\dddot{e}_1$:

$$
\begin{aligned}
\dddot{e}_1 &= (y-\gamma)^{(3)} = \dddot{y} - \dddot{\gamma} = \dddot{x}_1 - \dddot{\gamma} = \dot{x}_3 - \dddot{\gamma}\\
&= u - \dddot{\gamma} = -(\lambda_1+\lambda_2+\lambda_3)\ddot{e}_1 - (\lambda_1\lambda_2+\lambda_1\lambda_3+\\
&\quad \lambda_2\lambda_3)\dot{e}_1 - \lambda_1\lambda_2\lambda_3 e_1.
\end{aligned}
$$

Following the above deduction, (11) is reformulated as the following vector-matrix form:

$$
\dot{\mathbf{z}} = \begin{pmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\end{pmatrix} = Z\begin{pmatrix}z_1\\ z_2\\ z_3\end{pmatrix} = Z\mathbf{z}, \tag{12}
$$

where

$$
Z = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -\lambda_1\lambda_2\lambda_3 & -(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3) & -(\lambda_1+\lambda_2+\lambda_3)\end{pmatrix}.
$$

The characteristic polynomials of A_3 and Z are calculated as follows:

$$
\begin{aligned}
|\iota I - A_3| &= \begin{vmatrix}\iota+\lambda_1 & -1 & 0\\ 0 & \iota+\lambda_2 & -1\\ 0 & 0 & \iota+\lambda_3\end{vmatrix}\\
&= (\iota+\lambda_1)(\iota+\lambda_2)(\iota+\lambda_3)
\end{aligned}
$$

and

$$
\begin{aligned}
|\iota I - Z| &= \begin{vmatrix}\iota & -1 & 0\\ 0 & \iota & -1\\ \lambda_1\lambda_2\lambda_3 & (\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3) & \iota+(\lambda_1+\lambda_2+\lambda_3)\end{vmatrix}\\
&= \iota^3 + (\lambda_1+\lambda_2+\lambda_3)\iota^2 + (\lambda_1\lambda_2+\lambda_1\lambda_3+\\
&\quad \lambda_2\lambda_3)\iota + \lambda_1\lambda_2\lambda_3\\
&= (\iota+\lambda_1)(\iota+\lambda_2)(\iota+\lambda_3),
\end{aligned}
$$

respectively. Evidently, A_3 and Z have the same characteristic roots, i.e., $\iota_1 = -\lambda_1$, $\iota_2 = -\lambda_2$, and $\iota_3 = -\lambda_3$. It is still not sure that A_3 is similar to Z. We discuss the similarity in three cases.

1) Assume that λ_1, λ_2, and λ_3 are not equal to each other. Then, both A_3 and Z can be decomposed as [14]:

$$
A_3 = V_{A_3}^{-1}\begin{pmatrix}-\lambda_1 & 0 & 0\\ 0 & -\lambda_2 & 0\\ 0 & 0 & -\lambda_3\end{pmatrix}V_{A_3} = V_{A_3}^{-1} D V_{A_3} \tag{13}
$$

and

$$
Z = V_Z^{-1}\begin{pmatrix}-\lambda_1 & 0 & 0\\ 0 & -\lambda_2 & 0\\ 0 & 0 & -\lambda_3\end{pmatrix}V_Z = V_Z^{-1} D V_Z, \tag{14}
$$

where V_{A_3} and V_Z denote invertible matrices, and $V_{A_3}^{-1}$ and V_Z^{-1} correspond to the inverses of V_{A_3} and V_Z, respectively. By substituting $D = V_{A_3} A_3 V_{A_3}^{-1}$ into (14), the following equation is obtained:

$$
\begin{aligned}
Z &= V_Z^{-1} D V_Z = V_Z^{-1}(V_{A_3} A_3 V_{A_3}^{-1})V_Z\\
&= (V_{A_3}^{-1} V_Z)^{-1} A_3 (V_{A_3}^{-1} V_Z),
\end{aligned}
$$

which proves that A_3 is similar to Z.

2) Assume that two of λ_1, λ_2, and λ_3 are equal and the other one is not equal to them. Without loss of generality, we only consider the situation that $\lambda_1 = \lambda_2 = \lambda \neq \lambda_3$. During

Table 1: Ranks Needed for Calculating Jordan Canonical Form with $\lambda_1 = \lambda_2 = \lambda \neq \lambda_3$

	rank(X^0)	rank(X^1)	rank(X^2)	rank(X^3)	rank(X^4)
$X = -\lambda I - A_3$	3	2	1	1	1
$X = -\lambda_3 I - A_3$	3	2	2	2	2
$X = -\lambda I - Z$	3	2	1	1	1
$X = -\lambda_3 I - Z$	3	2	2	2	2

the calculation for the geometric multiplicity of eigenvalues, it is discovered that $-\lambda_1 I - A_3$ and $-\lambda_1 I - Z$ are equivalent. The elementary transformation from $-\lambda_1 I - Z$ to $-\lambda_1 I - A_3$ is shown as follows:

$$
\begin{aligned}
&-\lambda_1 I - Z = \begin{pmatrix} -\lambda_1 & -1 & 0 \\ 0 & -\lambda_1 & -1 \\ \lambda_1\lambda_2\lambda_3 & \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3 & \lambda_2+\lambda_3 \end{pmatrix} \\
&\xrightarrow{\mathbf{c}_1+\mathbf{c}_2\times(-\lambda_1)} \begin{pmatrix} 0 & -1 & 0 \\ \lambda_1^2 & -\lambda_1 & -1 \\ -\lambda_1^2(\lambda_2+\lambda_3) & \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3 & \lambda_2+\lambda_3 \end{pmatrix} \\
&\xrightarrow{\mathbf{r}_3+\mathbf{r}_2\times(\lambda_2+\lambda_3),\ \mathbf{c}_1+\mathbf{c}_3\times\lambda_1^2,\ \mathbf{c}_2+\mathbf{c}_3\times(-\lambda_2)} \begin{pmatrix} 0 & -1 & 0 \\ 0 & -\lambda_1+\lambda_2 & -1 \\ 0 & \lambda_2\lambda_3 & 0 \end{pmatrix} \\
&\xrightarrow{\mathbf{r}_3+\mathbf{r}_1\times\lambda_2\times\lambda_3} \begin{pmatrix} 0 & -1 & 0 \\ 0 & -\lambda_1+\lambda_2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \\
&\xrightarrow{\mathbf{r}_3+\mathbf{r}_2\times(\lambda_1-\lambda_3)+\mathbf{r}_1\times(\lambda_1-\lambda_2)\times(\lambda_1-\lambda_3)} \begin{pmatrix} 0 & -1 & 0 \\ 0 & -\lambda_1+\lambda_2 & -1 \\ 0 & 0 & -\lambda_1+\lambda_3 \end{pmatrix} \\
&= -\lambda_1 I - A_3,
\end{aligned}
$$

where $\mathbf{r}_i$ and $\mathbf{c}_j$ denote the ith row and jth column of the matrix, with $i = 1, 2, 3$ and $j = 1, 2, 3$. After calculation, $-\lambda_2 I - A_3$ and $-\lambda_3 I - A_3$ are equivalent to $-\lambda_2 I - Z$ and $-\lambda_3 I - Z$, respectively. It follows that equation (13) holds true if and only if equation (14) holds true. Thus, the next step is to determine the ranks of $-\lambda_1 I - A_3$, $-\lambda_2 I - A_3$, and $-\lambda_3 I - A_3$. With $\lambda_1 = \lambda_2 = \lambda \neq \lambda_3$, we obtain the ranks of three matrices as below:

$$
\begin{aligned}
&\text{rank}(-\lambda_1 I - A_3) = \text{rank}(-\lambda_2 I - A_3) = \text{rank}(-\lambda I - A_3) \\
&= \text{rank}\left(\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -\lambda+\lambda_3 \end{pmatrix}\right) = 2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\text{rank}(-\lambda_3 I - A_3) = \text{rank}\left(\begin{pmatrix} \lambda-\lambda_3 & -1 & 0 \\ 0 & \lambda-\lambda_3 & -1 \\ 0 & 0 & 0 \end{pmatrix}\right) \\
&= 2.
\end{aligned}
$$

Since the geometric multiplicity of $-\lambda$ (i.e., $3-\text{rank}(-\lambda I - A_3) = 1$) is less than its algebraic multiplicity (i.e., 2), A_3 cannot be decomposed as (13). Correspondingly, Z also cannot be decomposed as (14). Considering that any square matrix M can be decomposed as $M = V^{-1}JV$, where V denotes an invertible matrix and V^{-1} corresponds to the inverse of V, and J denotes the Jordan canonical form of M, we try to figure out whether A_3 is similar to Z by means of Jordan decomposition [15]. To obtain the Jordan canonical forms of A_3 and Z, we calculate some ranks of matrices and list them in Table 1. With the aid of Table 1, it is found out that the Jordan canonical form of A_3 is the same as that of Z, which is described as

$$
J = \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda_3 \end{pmatrix}.
$$

Thereby, it is easy to obtain $Z = (V_{A_3}^{-1}V_Z)^{-1}A_3(V_{A_3}^{-1}V_Z)$, and the proof, that A_3 is similar to Z in the case of $\lambda_1 = \lambda_2 = \lambda \neq \lambda_3$, is finished. Due to space limitation and the same proof method, the other two situations are omitted here.

3) Assume that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$. Consequently, we have

$$
-\lambda I - A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix},
$$

of which the rank is 2. Since the algebraic multiplicity of the eigenvalue $-\lambda$ is 3, the matrix A_3 cannot be decomposed as the form of (13). With the assistance of Table 2, it is verified that A_3 and Z have the same Jordan canonical form again, i.e.,

$$
J = \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{pmatrix} = A_3.
$$

Therefore, we prove that A_3 is similar to Z, under the condition that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$.

In summary, for any $\lambda_1 \in \mathbb{R}$, $\lambda_2 \in \mathbb{R}$, and $\lambda_3 \in \mathbb{R}$, the system (7) is equivalent to the system (12). Hence, the ZDC (6) is consistent with the design procedure.

4 Theoretical Analyses of TIS Equipped with ZDC

In this section, we analyze the stability and the convergence of the TIS with the control input (6) for tracking control.

Assume that the reference trajectory γ is smooth and bounded. Then, it is evident that the ZDC (6) is smooth and bounded. In accordance with the stability theory of systems [16], the TIS synthesized by the ZDC (6) is stable and convergent if the errors e_1, e_2, and e_3 are bounded and convergent.

Before proving the stability and the convergence of the TIS with the control input (6), we list two important lemmas as follows.

Lemma 1. *Assume that the impulse response of a linear time-invariant (LTI) causal system is $h = \exp(-\alpha_1 t)$, with $\alpha_1 \in \mathbb{R}^+$, and the input causal signal is $\exp(-\alpha_2 t)$, with $\alpha_2 \in \mathbb{R}^+$. Then, the output y of the system converges to zero globally and exponentially.*

Table 2: Ranks Needed for Calculating Jordan Canonical Form with $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$

	rank(X^0)	rank(X^1)	rank(X^2)	rank(X^3)	rank(X^4)	rank(X^5)
$X = -\lambda I - A_3$	3	2	1	0	0	0
$X = -\lambda I - Z$	3	2	1	0	0	0

proof: According to the theory of signals and systems [16], we have

$$\begin{aligned} y &= \exp(-\alpha_1 t) * \exp(-\alpha_2 t) \\ &= \int_0^t \exp(-\alpha_1 \tau)\exp(-\alpha_2(t-\tau))d\tau, \text{ for } t \geq 0, \end{aligned} \tag{15}$$

where the symbol $*$ denotes the convolution operator. Derived from equation (15), we have

$$\begin{cases} y = t\exp(-\alpha_1 t), \text{ with } \alpha_1 = \alpha_2, \\ y = \frac{\exp(-\alpha_1 t) - \exp(-\alpha_2 t)}{\alpha_2 - \alpha_1}, \text{ with } \alpha_1 \neq \alpha_2. \end{cases} \tag{16}$$

As t approaches infinity, the limit of equation group (16) is

$$\lim_{t\to\infty} y = 0.$$

Thereby, the output y of the system converges to zero globally and exponentially. The proof is completed. ■

Lemma 2. *Assume that the impulse response of a LTI causal system is $h = \exp(-\beta t)$, with $\beta \in \mathbb{R}^+$, and the input causal signal g is bounded with $|g| \leq g_\mathrm{m}$. Then, the absolute value of the output $|y|$ of the system converges to an error bound g_m/β globally and exponentially.*

proof: According to the theory of signals and systems [16], we have

$$\begin{aligned} y = g * h &= \int_0^t g(\tau)h(t-\tau)d\tau \\ &= \int_0^t g(\tau)\exp(-\beta(t-\tau))d\tau, \text{ for } t \geq 0. \end{aligned}$$

In consideration of the bounded g, an inequality is obtained as

$$|y| \leq \int_0^t g_\mathrm{m}\exp(-\beta(t-\tau))d\tau = \frac{g_\mathrm{m}(1-\exp(-\beta t))}{\beta} \leq \frac{g_\mathrm{m}}{\beta}. \tag{17}$$

Therefore, the absolute value of the output $|y|$ of the system converges to the error bound g_m/β globally and exponentially. The proof is completed. ■

During our investigation for the stability and the convergence of the TIS with the control input (6), we discover many significant properties and summarize them as the following Theorem 1, Corollary 1, Theorem 2, and Corollary 2.

Theorem 1. *For a smooth and bounded reference trajectory γ, starting from any initial state vector $\mathbf{x}_0 = [x_1(0), x_2(0), x_3(0)]^\mathrm{T} \in \mathbb{R}^3$, the noise-free TIS (3) equipped with the ZDC (6) is globally stable, and the tracking error e_1 converges to zero globally and exponentially.*

proof: The eigenvalues of A_3 in equation (8) are $-\lambda_1$, $-\lambda_2$, and $-\lambda_3$, and they lie on the left half of the complex plane. Hence, A_3 is a Hurwitz matrix and the noise-free TIS (3) with the control input (6) is globally stable. The solution of equation (8) is

$$\mathbf{e} = \exp(A_3 t)\mathbf{e}_0,$$

where $\mathbf{e}_0 = [e_1(0), e_2(0), e_3(0)]^\mathrm{T}$. Thus, for any initial state vector $\mathbf{x}_0$, the error vector $\mathbf{e}$ converges to a zero vector globally and exponentially. Therefore, the tracking error e_1 converges to zero globally and exponentially. The proof is completed. ■

Based on the proof of Theorem 1, we have the following corollary.

Corollary 1. *For a smooth and bounded reference trajectory γ, starting from any initial state vector $\mathbf{x}_0 = [x_1(0), x_2(0), x_3(0)]^\mathrm{T} \in \mathbb{R}^3$, the noise-free TIS (3) equipped with the ZDC (6) is globally stable, and the error vector $\mathbf{e}$ converges to a zero vector globally and exponentially.*

Theorem 2. *For a smooth and bounded reference trajectory γ, and a bounded noise g with $|g| \leq g_\mathrm{m}$, starting from any initial state vector $\mathbf{x}_0 = [x_1(0), x_2(0), x_3(0)]^\mathrm{T} \in \mathbb{R}^3$, the disturbed TIS (4) equipped with the ZDC (6) is globally stable, and the absolute value of the tracking error $|e_1|$ converges to an error bound $g_\mathrm{m}/(\lambda_1\lambda_2\lambda_3)$ globally and exponentially.*

proof: We rewrite equation (9) as the following equations:

$$\dot{e}_1 + \lambda_1 e_1 = e_2, \tag{18}$$

$$\dot{e}_2 + \lambda_2 e_2 = e_3, \tag{19}$$

$$\dot{e}_3 + \lambda_3 e_3 = g. \tag{20}$$

The solution of equation (20) is

$$e_3 = a_3\exp(-\lambda_3 t) + g * \exp(-\lambda_3 t), \tag{21}$$

where a_3 is a real constant. In view of equation (21), as t approaches infinity, the first term on the right side converges to zero globally and exponentially. In light of Lemma 2, the second term converges to an error bound g_m/λ_3 globally and exponentially. Consequently, $|e_3|$ converges to the error bound g_m/λ_3 globally and exponentially.

The solution of equation (19) is obtained as

$$e_2 = a_2\exp(-\lambda_2 t) + e_3 * \exp(-\lambda_2 t), \tag{22}$$

where a_2 is a real constant. By substituting equation (21) into equation (22), we obtain

$$\begin{aligned} e_2 =& a_2\exp(-\lambda_2 t) + a_3\exp(-\lambda_2 t) * \exp(-\lambda_3 t) + \\ & g * \exp(-\lambda_2 t) * \exp(-\lambda_3 t). \end{aligned} \tag{23}$$

In view of equation (23), as t approaches infinity, the first term on the right side converges to zero globally and exponentially. According to Lemma 1, the second term converges to zero globally and exponentially. For the third term, based on Lemma 2, we have

$$\begin{aligned} &|g * \exp(-\lambda_2 t) * \exp(-\lambda_3 t)| \leq \big| |g * \exp(-\lambda_2 t)| * \\ &\exp(-\lambda_3 t)\big| \leq \left|\frac{g_\mathrm{m}}{\lambda_2} * \exp(-\lambda_3 t)\right| \leq \frac{g_\mathrm{m}}{\lambda_2\lambda_3}. \end{aligned} \tag{24}$$

Thus, $|e_2|$ converges to an error bound $g_{\mathrm{m}}/(\lambda_2\lambda_3)$ globally and exponentially.

The solution of equation (18) is

$$e_1 = a_1\exp(-\lambda_1 t) + e_2 * \exp(-\lambda_1 t), \tag{25}$$

where a_1 is a real constant. By substituting equation (23) into equation (25), we obtain

$$\begin{aligned} e_1 =& a_1\exp(-\lambda_1 t) + a_2\exp(-\lambda_1 t) * \exp(-\lambda_2 t) + \\ & a_3\exp(-\lambda_1 t) * \exp(-\lambda_2 t) * \exp(-\lambda_3 t) + \\ & g * \exp(-\lambda_1 t) * \exp(-\lambda_2 t) * \exp(-\lambda_3 t). \end{aligned} \tag{26}$$

Based on (16), it follows that

$$\begin{aligned} & \exp(-\lambda_1 t) * \exp(-\lambda_2 t) * \exp(-\lambda_3 t) \\ =& \begin{cases} (t\exp(-\lambda_1 t)) * \exp(-\lambda_3 t), \text{ with } \lambda_1 = \lambda_2, \\ \frac{\exp(-\lambda_1 t) - \exp(-\lambda_2 t)}{\lambda_2 - \lambda_1} * \exp(-\lambda_3 t), \text{ with } \lambda_1 \neq \lambda_2. \end{cases} \\ =& \begin{cases} \frac{t^2\exp(-\lambda_3 t)}{2}, \text{ with } \lambda_1 = \lambda_2 = \lambda_3, \\ \frac{t\exp(-\lambda_1 t)}{\lambda_3 - \lambda_1} - \frac{\exp(-\lambda_1 t) - \exp(-\lambda_3 t)}{(\lambda_3 - \lambda_1)^2}, \text{with } \lambda_1 = \lambda_2 \neq \lambda_3, \\ \frac{\exp(-\lambda_1 t) * \exp(-\lambda_3 t) - \exp(-\lambda_2 t) * \exp(-\lambda_3 t)}{\lambda_2 - \lambda_1}, \text{with } \lambda_1 \neq \lambda_2. \end{cases} \end{aligned} \tag{27}$$

Evidently, as t approaches infinity, the first term on the right side of equation (26) converges to zero globally and exponentially. In term of Lemma 1, the second term and the third one, i.e., $a_3\times$(27), converge to zero globally and exponentially. In the light of Lemma 2 and (24), the fourth term converges to the error bound $g_{\mathrm{m}}/(\lambda_1\lambda_2\lambda_3)$ globally and exponentially. Thereby, for any initial state vector $\mathbf{x}_0$, the absolute value of the tracking error $|e_1|$ converges to the error bound $g_{\mathrm{m}}/(\lambda_1\lambda_2\lambda_3)$ globally and exponentially. The proof is completed. ■

Derived from the proof of Theorem 2, we have the following corollary.

Corollary 2. *For a smooth and bounded reference trajectory γ, and a bounded noise g with $|g| \leq g_{\mathrm{m}}$, starting from any initial state vector $\mathbf{x}_0 = [x_1(0), x_2(0), x_3(0)]^{\mathrm{T}} \in \mathbb{R}^3$, the disturbed TIS (4) equipped with the ZDC (6) is globally stable, and the vector containing the absolute values of the errors e_1, e_2, and e_3, i.e., $[|e_1|, |e_2|, |e_3|]^{\mathrm{T}}$, converges to an error-bound vector $[g_{\mathrm{m}}/(\lambda_1\lambda_2\lambda_3), g_{\mathrm{m}}/(\lambda_2\lambda_3), g_{\mathrm{m}}/\lambda_3]^{\mathrm{T}}$ globally and exponentially.*

5 Theoretical Analyses of MIS Equipped with ZDC

In this section, we generalize the ZDC for the MIS, and investigate the error dynamics of the MIS synthesized by the ZDC. Besides, we explore the stability and the convergence of the MIS synthesized by the ZDC.

At first, we design the ZDC for the MIS. By adopting the ZD method [2], the ZDC for the MIS (i.e., the nth-order integrator system) is formulated as

$$u = \gamma^{(n)} - K^{\mathrm{T}}E, \tag{28}$$

where $E = [e_1, \dot{e}_1, \cdots, e_1^{(n-1)}]^{\mathrm{T}} \in \mathbb{R}^n$, $K = [k_0, k_1, \cdots, k_{n-1}]^{\mathrm{T}} \in \mathbb{R}^n$, and $\gamma^{(n)}$ is the nth-order derivative of γ. Moreover, K satisfies the condition that all the roots $\{-\lambda_i\}$ of the polynomial $p(s) = \prod_{i=1}^{n}(s+\lambda_i) = s^n + k_{n-1}s^{n-1} + \cdots + k_0$ lie on the left half of the complex plane, which actually lie on the left-half axis of the complex plane.

The state equations of e_1, e_2, $\cdots$, and e_n of the noise-free MIS (1) synthesized by the ZDC (28) are presented as

$$\begin{aligned} \dot{\mathbf{e}} =& \begin{pmatrix} -\lambda_1 & 1 & & & & \\ & -\lambda_2 & 1 & & \mathbf{0} & \\ & & -\lambda_3 & \ddots & & \\ & & & \ddots & 1 & \\ & \mathbf{0} & & & -\lambda_{n-1} & 1 \\ & & & & & -\lambda_n \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_{n-1} \\ e_n \end{pmatrix} \\ =& A_n\mathbf{e}. \end{aligned} \tag{29}$$

The state equations of e_1, e_2, $\cdots$, and e_n of the disturbed MIS (2) with the control input (28) are formulated as

$$\dot{\mathbf{e}} = A_n\mathbf{e} + \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} g = A_n\mathbf{e} + B_n g. \tag{30}$$

We summarize the significant properties of the MIS with the control input (28) as the following Theorem 3, Corollary3, Theorem 4, and Corollary 4.

Theorem 3. *For a smooth and bounded reference trajectory γ, starting from any initial state vector $\mathbf{x}_0 = [x_1(0), x_2(0), \cdots, x_n(0)]^{\mathrm{T}} \in \mathbb{R}^n$, the noise-free MIS (1) equipped with ZDC (28) is globally stable, and the tracking error e_1 converges to zero globally and exponentially.*

proof: The eigenvalues of A_n are denoted by $\{-\lambda_i\}$ with $i = 1, 2, \cdots, n$, which lie on the left half of the complex plane. Hence, A_n is a Hurwitz matrix and the noise-free MIS (1) synthesized by the ZDC (28) is globally stable. The solution of (29) is

$$\mathbf{e} = \exp(A_n t)\mathbf{e}_0,$$

where $\mathbf{e}_0 = [e_1(0), e_2(0), \cdots, e_n(0)]^{\mathrm{T}}$. For any initial state vector $\mathbf{x}_0$, the error vector $\mathbf{e}$ converges to zero globally and exponentially. Hence, the tracking error e_1 converges to zero globally and exponentially. The proof is completed. ■

In light of the above proof, we have the following corollary.

Corollary 3. *For a smooth and bounded reference trajectory γ, starting from any initial state vector $\mathbf{x}_0 = [x_1(0), x_2(0), \cdots, x_n(0)]^{\mathrm{T}} \in \mathbb{R}^n$, the noise-free MIS (1) equipped with the ZDC (28) is globally stable, and the error vector $\mathbf{e}$ converges to a zero vector globally and exponentially.*

Theorem 4. *For a smooth and bounded reference trajectory γ, and a bounded noise g with $|g| \leq g_{\mathrm{m}}$, starting from any initial state vector $\mathbf{x}_0 = [x_1(0), x_2(0), \cdots, x_n(0)]^{\mathrm{T}} \in \mathbb{R}^n$, the disturbed MIS (2) equipped with the ZDC (28) is globally stable, and the absolute value of the tracking error $|e_1|$ converges to an error bound $g_{\mathrm{m}}/\prod_{i=1}^{n}\lambda_i$ globally and exponentially.*

proof: The solution of equation (30) is

$$\begin{cases} e_n = \alpha_n \exp(-\lambda_n t) + g * \exp(-\lambda_n t) \\ e_{n-1} = \alpha_{n-1} \exp(-\lambda_{n-1} t) + e_n * \exp(-\lambda_{n-1} t) \\ \quad \vdots \\ e_1 = \alpha_1 \exp(-\lambda_1 t) + e_2 * \exp(-\lambda_1 t), \end{cases}$$

where α_i, with $i = 1, 2, \cdots, n$, are real constants. Further, we have

$$\begin{aligned} e_1 =& \alpha_1 \exp(-\lambda_1 t) + \alpha_2 \exp(-\lambda_1 t) * \exp(-\lambda_2 t) + \cdots + \\ & \alpha_n \exp(-\lambda_1 t) * \exp(-\lambda_2 t) * \cdots * \exp(-\lambda_n t) + \\ & g * \exp(-\lambda_1 t) * \exp(-\lambda_2 t) * \cdots * \exp(-\lambda_n t). \end{aligned} \quad (31)$$

As t approaches infinity, the first term on the right side of equation (31) converges to zero globally and exponentially. Generalized from Lemma 1 and (27), the second term through the nth one converge to zero globally and exponentially. Generalized from Lemma 2 and (24), the last term converges to the error bound $g_{\mathrm{m}}/(\lambda_1 \lambda_2 \cdots \lambda_n)$, that is, $g_{\mathrm{m}}/\prod_{i=1}^{n} \lambda_i$, globally and exponentially. Thereby, for any initial state vector $\mathbf{x}_0$, the absolute value of the tracking error $|e_1|$ converges to the error bound $g_{\mathrm{m}}/\prod_{i=1}^{n} \lambda_i$ globally and exponentially. The proof is completed. ■

Generalized from the proof of Lemma 2, Theorem 2, and Theorem 4, we have the following corollary.

Corollary 4. *For a smooth and bounded reference trajectory γ, and a bounded noise g with $|g| \leq g_{\mathrm{m}}$, starting from any initial state vector $\mathbf{x}_0 = [x_1(0), x_2(0), \cdots, x_n(0)]^{\mathrm{T}} \in \mathbb{R}^n$, the disturbed MIS (2) synthesized by the ZDC (28) is globally stable, and the vector containing the absolute values of the errors e_1, e_2, $\cdots$, and e_n, i.e., $[|e_1|, |e_2|, \cdots, |e_n|]^{\mathrm{T}}$, converges to an error-bound vector $[g_{\mathrm{m}}/\prod_{i=1}^{n} \lambda_i, g_{\mathrm{m}}/\prod_{i=2}^{n} \lambda_i, \cdots, g_{\mathrm{m}}/\lambda_n]^{\mathrm{T}}$ globally and exponentially.*

6 Conclusion

In this paper, ZD has been applied to designing the ZDC for the noise-free MIS (1) and the disturbed MIS (2) with bounded noise pollution. Via the design procedure of the ZDC for tracking control, the stability of the MIS is converted into the stability of its error dynamics, which comprises the state equations of the tracking error and other ZD errors. Moreover, the theoretical analyses given in Section 4 and Section 5 have proved the stability and the convergence of the MIS equipped with the ZDC, which means that regardless of noise pollution, its tracking error and other ZD errors are bounded and convergent. In addition, we have discovered a system equivalent to the error dynamics of the noise-free TIS synthesized by the ZDC.

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References

[1] Y. Zhang, T. Qiao, D. Zhang, H. Tan, and D. Liang, Simple effective Zhang-dynamics stabilization control of the 4th-order hyper-chaotic Lu system with one input, in *Proceedings of International Conference on Natural Computation, Fuzzy Systems and Knowledge Discovery*, 2016: 325–330.

[2] C. Chen, Y. Ling, D. Zhang, N. Shi, and Y. Zhang, Noisy Zhang-dynamics (ZD) method for Genesio chaotic (GC) system synchronization: Elegant analyses and unequal-parameter extension, in *Proceedings of IEEE Symposium Series on Computational Intelligence*, 2019: 482–487.

[3] Y. Zhang, F. Luo, X. Yu, J. Liu, and J. Chen, A continuous-time model and simulative verifications of Zhang-gradient type Zhang reciprocal for time-varying numbers, in *Proceedings of International Conference on Natural Computation*, 2013: 750–754.

[4] Y. Zhang, W. Li, D. Guo, Z. Zhang, and S. Fu, Feedback-type MWVN scheme and its acceleration-level equivalent scheme proved by Zhang dynamics, in *Proceedings of International Conference on Control Engineering and Communication Technology*, 2012: 180–183.

[5] Y. Zhang, D. Zhang, B. Qiu, J. Wang, and J. Li, Sigmoid function aided Zhang dynamics control for output tracking of time-varying linear system with bounded input, in *Proceedings of Chinese Control Conference*, 2016: 5901–5905.

[6] Y. Zhang, H. Xiao, D. Chen, S. Ding, and P. Chen, Link from ZD control to Pascal's triangle illustrated via multiple-integrator systems, in *Proceedings of International Conference on Natural Computation, Fuzzy Systems and Knowledge Discovery*, 2016: 2127–2131.

[7] B. Zhou and X. Yang, Global stabilization of the multiple integrators system by delayed and bounded controls, *IEEE Transactions on Automatic Control*, 61(12): 4222–4228, 2016.

[8] S. Ding and Z. Xing, Some results on stabilization of multiple integrators with input saturation, in *Proceedings of Chinese Control Conference*, 2013: 1538–1543.

[9] C. Hu, X. Kang, and Y. Zhang, Singularity-conquering Zhang-gradient controller groups for tracking control of Brockett integrator, in *Proceedings of Chinese Control and Decision Conference*, 2018: 6291–6296.

[10] H. M. Becerra, C. R. Vzquez, G. Arechavaleta, and J. Delfin, Predefined-time convergence control for high-order integrator systems using time base generators, *IEEE Transactions on Control Systems Technology*, 26(5): 1866–1873, 2018.

[11] Y. Zhang, K. Zhai, Y. Wang, D. Chen, and C. Peng, Design and illustration of ZG controllers for linear and nonlinear tracking control of double-integrator system, in *Proceedings of Chinese Control Conference*, 2014: 3462–3467.

[12] Y. Zhang, S. Ding, D. Chen, M. Mao, and K. Zhai, Zhang-gradient controllers for tracking control of multiple-integrator systems, *ASME Journal of Dynamic Systems Measurement and Control*, 137(11): 111013, 2015.

[13] V. G. Rao and D. S. Bernstein, Naive control of the double integrator: A comparison of a dozen diverse controllers under off-nominal conditions, in *Proceedings of American Control Conference*, 1999: 1477–1481.

[14] S. J. Leon, I. Bica, and T. Hohn, *Linear Algebra with Applications*. New Jersey: Pearson Prentice Hall, 2006.

[15] N. Jacobson, *Basic Algebra I*. Massachusetts: Courier Corporation, 2012.

[16] A. V. Oppenheim, A. S. Willsky, and S. H. Nawab, *Signals and Systems*. New Jersey: Prentice Hall, 1997.