No 8-Instant ZTD (Zhang Time Discretization) Formula with Quintic Precision or Higher as Proved

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***Abstract*—Since 2014, Zhang time discretization (ZTD, also termed Zhang *et al.* discretization) formulas have been put forward as a new method for time discretization by Zhang *et al*. Originally, ZTD formulas were used to obtain discrete Zhang neural network (ZNN) model from continuous ZNN one. ZTD formulas were also used later to discretize other continuous-time systems. So far, ZTD formulas with various number of instants have been proposed and applied, and the highest precisions of the 2-instant ZTD formula to 7-instant ZTD formulas have been discovered. Instinctively, we want to find out the highest precision of 8-instant ZTD formulas. During our investigation, we ascertain that any convergent 8-instant ZTD formula cannot possess quintic precision or higher. Combined with previous work, we conclude that any 8-instant ZTD formula converges with a truncation error in proportion to the 4-th power of the sampling interval or greater. The conclusion is proved in our paper.**

***Index Terms*—Zhang time discretization formulas, Taylor expansions, Tustin transform, Routh stability criterion, Jury table**

I. Introduction

From quantities of scientific researches and practical applications [1]–[3], we find out that it is difficult to solve a certain discrete-time problem by adopting a continuous-time model on a digital computer. However, as digital computers and technologies are applied more widely, it is becoming increasingly important to solve discrete-time problems. Therefore, it is inevitable for scientists to find some methods to discretize continuous-time models. Moreover, the basic point is that the discrete-time model which is attained from continuous-time one should be stable, convergent, and accurate [4]–[9].

In the field of the time discretization of continuous models, Euler forward formula (EFF, also termed Euler forward difference formula) concentrates on approaching the 1-st order time derivative quite simply and relatively roughly. So do Zhang time discretization (ZTD) formulas [4]–[6], but more accurately. Since 2014, ZTD formulas have been proposed as a novel time-discretization method by Zhang *et al.* [4]–[9]. Originally, ZTD formulas were used to discretize Zhang neural network (ZNN) models and Zhang dynamics (ZD) models [10], [11]. Besides, ZNN and ZD, playing a powerful role in the time-varying (also termed time-dependent) field, have been improved frequently over the twenty years [12]–[16] . Therefore, the effective time discretization for ZNN models and ZD models is fairly important. It is worth noting that by adopting ZTD formulas to discretize neural-dynamic systems and also other ordinary derivative equation systems, the obtained discrete-time model is stable, convergent, and accurate [4], [17]–[19]. ZTD formulas approach the 1-st order derivative of $\phi(t_k)$ with regard to time, i.e., $\dot{\phi}(t_k)$, by using $\phi(t_{k+1})$, $\phi(t_k)$, and $\phi(t_{k-1})$ or more. The number of instants influences the precision of the approximation. According to previous work [5], [6], [8], [9], [20]–[29], ZTD formulas with various numbers of instants have been proposed and applied in the related researches. For example, EFF can be considered as a convergent ZTD formula with 2 instants. Its truncation error is in proportion to the sampling interval, i.e., $O(\tau)$, where τ represents the sampling interval, i.e., the time step-size. By Taylor expansions, we can get convergent 3-instant ZTD formulas with the same truncation error as that of EFF, convergent 4-instant ZTD formulas with a smaller truncation error being $O(\tau^2)$, convergent 5-instant and 6-instant ZTD formulas with the same truncation error being $O(\tau^3)$, convergent 7-instant and 8-instant ZTD formulas with the same truncation error being $O(\tau^4)$. By utilizing these convergent ZTD formulas, the discrete-time models derived from continuous-time models are stable, convergent, and accurate. That is the power of ZTD formulas.

In general, the more precise a ZTD formula is, the smaller the error of the corresponding discrete-time model is. Thus, we would always like to seek more precise ZTD formulas. During our exploration, we ascertain that there is no 8-instant ZTD formula having quintic or higher precision. In other words, the highest precision of any 8-instant ZTD formula is order 4. It is worth mentioning that convergent 8-instant ZTD formulas with a truncation error being $O(\tau^4)$ are derived by authors in [30], [31].

The remainder of our paper is divided into three parts. In the next section, the general n-instant ZTD formula is shown for convenience of the further research, and we sum up some ZTD formulas which have been proposed and applied, including 2-instant ZTD formula, 3-instant ZTD formula, 4-instant ZTD formula, 5-instant ZTD formula, 6-instant ZTD formula, 7-instant ZTD formula, and 8-instant ZTD formula. Then, in Section III, we prove that there is no 8-instant ZTD formula which converges with order 5 or higher. Finally, in Section IV, the conclusion is presented.

TABLE I
SPECIFIC ZTD FORMULAS WITH n INSTANTS, WHERE $n = 2, 3, 4, 5, 6, 7$, AND 8

n	Specific ZTD formula	References
2	$\dot{\phi}_k = \frac{\phi_{k+1}-\phi_k}{\tau} + O(\tau)$	[8], [9], [20]
3	$\dot{\phi}_k = \frac{2\phi_{k+1}-\phi_k-\phi_{k-1}}{3\tau} + O(\tau)$	[6]
4	$\dot{\phi}_k = \frac{2\phi_{k+1}-3\phi_k+2\phi_{k-1}-\phi_{k-2}}{6\tau} + O(\tau^2)$	[24]
5	$\dot{\phi}_k = \frac{57\phi_{k+1}+14\phi_k-54\phi_{k-1}-30\phi_{k-2}+13\phi_{k-3}}{132\tau} + O(\tau^3)$	[25]
6	$\dot{\phi}_k = \frac{23\phi_{k+1}-11\phi_k-4\phi_{k-1}-5\phi_{k-2}-7\phi_{k-3}+4\phi_{k-4}}{42\tau} + O(\tau^3)$	[28]
7	$\dot{\phi}_k = \frac{163\phi_{k+1}+81\phi_k-155\phi_{k-1}-160\phi_{k-2}+45\phi_{k-3}+47\phi_{k-4}-21\phi_{k-5}}{420\tau} + O(\tau^4)$	[28]
8	$\dot{\phi}_k = \frac{6000\phi_{k+1}-2593\phi_k-1800\phi_{k-1}+1200\phi_{k-2}-3600\phi_{k-3}-2475\phi_{k-4}+4968\phi_{k-5}-1700\phi_{k-6}}{11460\tau} + O(\tau^4)$	[30]

II. GENERAL n-INSTANT ZTD FORMULA AND SUMMARY ABOUT DISCOVERED ZTD FORMULAS

For convenience and for completeness, the general ZTD formula with n instants is shown in this section. According to the preceding work [5], [6], [8], [9], [20]–[31], the general ZTD formula with n instants is written as

$$\dot{\phi}_k = \frac{1}{\tau}\left(\sum_{i=1}^{n} a_{n-i+1}\phi_{k-i+2}\right) + O(\tau^p), \tag{1}$$

where $\tau \in (0,1)$ denotes the sampling interval (i.e., the time step-size); k denotes the index; a_i represents the corresponding coefficient; the function ϕ and its time derivatives (e.g., the 1-st order time derivative $\dot{\phi}$) are assumed to be continuous; ϕ_k represents the value of $\phi(t)$ at the instant $t_k = k\tau$, i.e., $\phi_k = \phi(t_k)$; $\dot{\phi}_k$ represents $\dot{\phi}(t_k)$ similarly; $O(\tau^p)$ denotes the truncation error being proportional to τ^p, i.e., of the order of τ^p.

Table I represents some specific ZTD formulas with n instants, where $n = 2, 3, 4, 5, 6, 7$, and 8. With $n = 2$, the ZTD formula is unique and it is known widely as EFF. As a 2-instant ZTD formula, it can discretize the neural-dynamic systems which solve time-varying matrix pseudoinversion, time-varying nonlinear optimization, and other time-varying problems [8], [9], [20] simply but less accurately. With $n = 3$, the highest precision of ZTD formulas is the same as the unique 2-instant ZTD formula's one, and they can discretize the continuous model designed for future minimization [6]. With $n = 4$, the highest precision of ZTD formulas is higher than the convergent 3-instant ZTD formulas' one. The convergent 4-instant ZTD formulas can discretize neural-dynamic systems which solve nonlinear equation systems, time-varying four fundamental operations, ordinary derivative equations, and so on [5], [20]–[22], [24]. With $n = 5$, the highest precision of ZTD formulas is higher than the convergent 4-instant ZTD formulas' one, and they can discretize the variants derived from the combination of ZNN and Getz-Marsden dynamic system, namely GMDS-ZNN variants, for time-varying matrix inversion on complex field and solve generalized Sylvester future matrix system [25], [26]. With $n = 6$, the highest precision of ZTD formulas is the same as the convergent 5-instant ZTD formulas' one, and, with $n = 7$, the highest precision is improved again. Both the convergent 6-instant ZTD formulas and 7-instant ZTD formulas can be applied to discretizing the model for the sling-load helicopter system with high effectiveness [27], [28]. With $n = 8$, some ZTD formulas with quartic precision are derived, and they can be applied in the online singular value decomposition of matrices varied with time, and the motion planning and control of redundant manipulators for time discretization [30], [31].

Evidently, ZTD formulas do well in the time discretization of continuous-time systems. With the increasing requirement of the error of the obtained discrete-time model, ZTD formulas with higher precision are demanded. Thus, we desire to discover ZTD formulas with higher precision or their interesting patterns. Besides, it hits us that the first task is to find the highest precision of 8-instant ZTD formulas.

III. NONEXISTENCE OF 8-INSTANT ZTD FORMULA WITH QUINTIC PRECISION OR HIGHER

Considering that 2-instant ZTD formula and 3-instant ZTD formulas have the same highest precision, 5-instant ZTD formulas and 6-instant ZTD formulas have the same highest precision, we may assume that 8-instant ZTD formulas have the same highest precision as 7-instant ZTD formulas' one. Out of curiosity, we try to prove the assumption in this paper. Besides, it is logical for us to seek more precise ZTD formulas. There is no doubt that one feasible method is to adopt more instants to improve the precision.

Based on the general ZTD formula with n instants (1), we get the general 8-instant (i.e., 7-step) ZTD formula readily:

$$\begin{aligned}\dot{\phi}_k = &\frac{a_8}{\tau}\phi_{k+1} + \frac{a_7}{\tau}\phi_k + \frac{a_6}{\tau}\phi_{k-1} + \frac{a_5}{\tau}\phi_{k-2} + \frac{a_4}{\tau}\phi_{k-3} \\ &+ \frac{a_3}{\tau}\phi_{k-4} + \frac{a_2}{\tau}\phi_{k-5} + \frac{a_1}{\tau}\phi_{k-6} + O(\tau^p).\end{aligned} \tag{2}$$

During our investigation for ZTD formulas with higher precision, we discover many important properties and we summarize them as the following theorems. Note that the theorems are actually the steps of the proof that any 8-instant

ZTD formula converges with a truncation error being $O(\tau^4)$ or greater.

Theorem 1: Assume that $\tau \in (0,1)$, and $O(\tau^7)$ corresponds to the 7th-order truncation error. Then, there is no convergent 8-instant ZTD formula with a truncation error being $O(\tau^7)$ or smaller.

Proof: In the light of the Appendix, we know that the general 8-instant ZTD formula (2) is convergent under the condition that (2) is consistent and zero-stable. Firstly, we should check its consistency. The 8th-order Taylor expansions of ϕ_{k+1}, ϕ_{k-1}, ϕ_{k-2}, ϕ_{k-3}, ϕ_{k-4}, ϕ_{k-5}, and ϕ_{k-6} are formulated as

$$\begin{aligned}\phi_{k+1} =&\phi_k + \tau\dot{\phi}_k + \frac{\tau^2}{2!}\ddot{\phi}_k + \frac{\tau^3}{3!}\phi_k^{(3)} + \frac{\tau^4}{4!}\phi_k^{(4)}\\ &+ \frac{\tau^5}{5!}\phi_k^{(5)} + \frac{\tau^6}{6!}\phi_k^{(6)} + \frac{\tau^7}{7!}\phi_k^{(7)} + O(\tau^8),\end{aligned} \tag{3}$$

$$\begin{aligned}\phi_{k-1} =&\phi_k - \tau\dot{\phi}_k + \frac{\tau^2}{2!}\ddot{\phi}_k - \frac{\tau^3}{3!}\phi_k^{(3)} + \frac{\tau^4}{4!}\phi_k^{(4)}\\ &- \frac{\tau^5}{5!}\phi_k^{(5)} + \frac{\tau^6}{6!}\phi_k^{(6)} - \frac{\tau^7}{7!}\phi_k^{(7)} + O(\tau^8),\end{aligned} \tag{4}$$

$$\begin{aligned}\phi_{k-2} =&\phi_k - 2\tau\dot{\phi}_k + \frac{(2\tau)^2}{2!}\ddot{\phi}_k - \frac{(2\tau)^3}{3!}\phi_k^{(3)} + \frac{(2\tau)^4}{4!}\phi_k^{(4)}\\ &- \frac{(2\tau)^5}{5!}\phi_k^{(5)} + \frac{(2\tau)^6}{6!}\phi_k^{(6)} - \frac{(2\tau)^7}{7!}\phi_k^{(7)} + O(\tau^8),\end{aligned} \tag{5}$$

$$\begin{aligned}\phi_{k-3} =&\phi_k - 3\tau\dot{\phi}_k + \frac{(3\tau)^2}{2!}\ddot{\phi}_k - \frac{(3\tau)^3}{3!}\phi_k^{(3)} + \frac{(3\tau)^4}{4!}\phi_k^{(4)}\\ &- \frac{(3\tau)^5}{5!}\phi_k^{(5)} + \frac{(3\tau)^6}{6!}\phi_k^{(6)} - \frac{(3\tau)^7}{7!}\phi_k^{(7)} + O(\tau^8),\end{aligned} \tag{6}$$

$$\begin{aligned}\phi_{k-4} =&\phi_k - 4\tau\dot{\phi}_k + \frac{(4\tau)^2}{2!}\ddot{\phi}_k - \frac{(4\tau)^3}{3!}\phi_k^{(3)} + \frac{(4\tau)^4}{4!}\phi_k^{(4)}\\ &- \frac{(4\tau)^5}{5!}\phi_k^{(5)} + \frac{(4\tau)^6}{6!}\phi_k^{(6)} - \frac{(4\tau)^7}{7!}\phi_k^{(7)} + O(\tau^8),\end{aligned} \tag{7}$$

$$\begin{aligned}\phi_{k-5} =&\phi_k - 5\tau\dot{\phi}_k + \frac{(5\tau)^2}{2!}\ddot{\phi}_k - \frac{(5\tau)^3}{3!}\phi_k^{(3)} + \frac{(5\tau)^4}{4!}\phi_k^{(4)}\\ &- \frac{(5\tau)^5}{5!}\phi_k^{(5)} + \frac{(5\tau)^6}{6!}\phi_k^{(6)} - \frac{(5\tau)^7}{7!}\phi_k^{(7)} + O(\tau^8),\end{aligned} \tag{8}$$

and

$$\begin{aligned}\phi_{k-6} =&\phi_k - 6\tau\dot{\phi}_k + \frac{(6\tau)^2}{2!}\ddot{\phi}_k - \frac{(6\tau)^3}{3!}\phi_k^{(3)} + \frac{(6\tau)^4}{4!}\phi_k^{(4)}\\ &- \frac{(6\tau)^5}{5!}\phi_k^{(5)} + \frac{(6\tau)^6}{6!}\phi_k^{(6)} - \frac{(6\tau)^7}{7!}\phi_k^{(7)} + O(\tau^8).\end{aligned} \tag{9}$$

By substituting ϕ_{k+1}, ϕ_{k-1}, ϕ_{k-2}, ϕ_{k-3}, ϕ_{k-4}, ϕ_{k-5}, and ϕ_{k-6} respectively of (3), (4), (5), (6), (7), (8), and (9) into (2), we obtain the following equation:

$$\begin{aligned}&b_0\phi_k + b_1\tau\dot{\phi}_k + b_2\tau^2\ddot{\phi}_k + b_3\tau^3\phi_k^{(3)} + b_4\tau^4\phi_k^{(4)} + b_5\tau^5\phi_k^{(5)}\\ &+ b_6\tau^6\phi_k^{(6)} + b_7\tau^7\phi_k^{(7)} + O(\tau^8) = O(\tau^{p+1}),\end{aligned} \tag{10}$$

where

$$\begin{aligned}b_0 =&\sum_{i=1}^{8} a_i,\\ b_1 =&- 6a_1 - 5a_2 - 4a_3 - 3a_4 - 2a_5 - a_6 + a_8 - 1,\\ b_2 =&18a_1 + 25a_2/2 + 8a_3 + 9a_4/2 + 2a_5 + a_6/2\\ &+ a_8/2,\\ b_3 =&- 36a_1 - 125a_2/6 - 32a_3/3 - 9a_4/2 - 4a_5/3\\ &- a_6/6 + a_8/6,\\ b_4 =&54a_1 + 625a_2/24 + 32a_3/3 + 27a_4/8 + 2a_5/3\\ &+ a_6/24 + a_8/24,\\ b_5 =&- 324a_1/5 - 625a_2/24 - 128a_3/15 - 81a_4/40\\ &- 4a_5/15 - a_6/120 + a_8/120,\\ b_6 =&324a_1/5 + 3125a_2/144 + 256a_3/45 + 81a_4/80\\ &+ 4a_5/45 + a_6/720 + a_8/720,\end{aligned}$$

and

$$\begin{aligned}b_7 =&- 1944a_1/35 - 15625a_2/1008 - 1024a_3/315\\ &- 243a_4/560 - 8a_5/315 - a_6/5040 + a_8/5040.\end{aligned}$$

If the general 8-instant ZTD formula (2) is consistent of order 7, then p in (2) should be equal to 7 with the following condition group to be satisfied:

$$\begin{cases}b_0 =\sum_{i=1}^{8} a_i = 0,\\ b_1 =- 6a_1 - 5a_2 - 4a_3 - 3a_4 - 2a_5 - a_6 + a_8 - 1\\ \quad=0,\\ b_2 =18a_1 + 25a_2/2 + 8a_3 + 9a_4/2 + 2a_5 + a_6/2\\ \quad + a_8/2 = 0,\\ b_3 =- 36a_1 - 125a_2/6 - 32a_3/3 - 9a_4/2 - 4a_5/3\\ \quad - a_6/6 + a_8/6 = 0,\\ b_4 =54a_1 + 625a_2/24 + 32a_3/3 + 27a_4/8 + 2a_5/3\\ \quad + a_6/24 + a_8/24 = 0,\\ b_5 =- 324a_1/5 - 625a_2/24 - 128a_3/15 - 81a_4/40\\ \quad - 4a_5/15 - a_6/120 + a_8/120 = 0,\\ b_6 =324a_1/5 + 3125a_2/144 + 256a_3/45 + 81a_4/80\\ \quad + 4a_5/45 + a_6/720 + a_8/720 = 0,\\ b_7 =- 1944a_1/35 - 15625a_2/1008 - 1024a_3/315\\ \quad - 243a_4/560 - 8a_5/315 - a_6/5040 + a_8/5040\\ \quad =0.\end{cases} \tag{11}$$

The solution of (11) is

$$\begin{cases} a_1 = 1/42, \quad a_2 = -1/5, \\ a_3 = 3/4, \quad a_4 = -5/3, \\ a_5 = 5/2, \quad a_6 = -3, \\ a_7 = 29/20, \quad a_8 = 1/7. \end{cases}$$

With the values of a_1 through a_8, we obtain the characteristic equation of (2) as

$$\begin{aligned} \rho(\lambda) = 60\lambda^7 + 609\lambda^6 - 1260\lambda^5 + 1050\lambda^4 - 700\lambda^3 \\ + 315\lambda^2 - 84\lambda + 10 = 0. \end{aligned} \tag{12}$$

The roots of equation (12) are $\lambda_1 = 12.0244$, $\lambda_2 = 0.3119$, $\lambda_3 = 1$, $\lambda_4 = 0.0127 + 0.5896\mathrm{i}$, $\lambda_5 = 0.0127 - 0.5896\mathrm{i}$, $\lambda_6 = 0.2685 + 0.2359\mathrm{i}$ and $\lambda_7 = 0.2685 - 0.2358\mathrm{i}$, with i denoting the imaginary unit. Evidently, λ_1 lies outside of the unit circle, which means that the 8-instant ZTD formula (2) is not zero-stable with coefficients $a_1 = 1/42$, $a_2 = -1/5$, $a_3 = 3/4$, $a_4 = -5/3$, $a_5 = 5/2$, $a_6 = -3$, $a_7 = 29/20$, and $a_8 = 1/7$. Therefore, there are no coefficients which guarantee that the general 8-instant ZTD formula (2) is convergent with a seventh order truncation error. Furthermore, it is evident that we cannot derive any convergent 8-instant ZTD formula on the basis of the 9-th order or higher order Taylor expansions. Thus, there is no convergent 8-instant ZTD formula with a truncation error being $O(\tau^7)$ or smaller. Here, The proof is completed. □

Theorem 2: Assume that $\tau \in (0, 1)$, and $O(\tau^6)$ corresponds to the 6th-order truncation error. Then, there is no convergent 8-instant ZTD formula with a truncation error being $O(\tau^6)$.

Proof: Provided that the general ZTD formula with 8 instants is consistent of order 6, then $p = 6$ in formula (2). In order to make it meet the consistency requirement, the following condition group should be satisfied:

$$\begin{cases} b_0 = \sum_{i=1}^{8} a_i = 0, \\ b_1 = -6a_1 - 5a_2 - 4a_3 - 3a_4 - 2a_5 - a_6 + a_8 - 1 \\ \quad = 0, \\ b_2 = 18a_1 + 25a_2/2 + 8a_3 + 9a_4/2 + 2a_5 + a_6/2 \\ \quad + a_8/2 = 0, \\ b_3 = -36a_1 - 125a_2/6 - 32a_3/3 - 9a_4/2 - 4a_5/3 \\ \quad - a_6/6 + a_8/6 = 0, \\ b_4 = 54a_1 + 625a_2/24 + 32a_3/3 + 27a_4/8 + 2a_5/3 \\ \quad + a_6/24 + a_8/24 = 0, \\ b_5 = -324a_1/5 - 625a_2/24 - 128a_3/15 - 81a_4/40 \\ \quad - 4a_5/15 - a_6/120 + a_8/120 = 0, \\ b_6 = 324a_1/5 + 3125a_2/144 + 256a_3/45 + 81a_4/80 \\ \quad + 4a_5/45 + a_6/720 + a_8/720 = 0. \end{cases} \tag{13}$$

By solving (13), we obtain the following solution:

$$\begin{cases} a_2 = -7a_1 - 1/30, \quad a_3 = 21a_1 + 1/4, \\ a_4 = -35a_1 - 5/6, \quad a_5 = 35a_1 + 5/3, \\ a_6 = -21a_1 - 5/2, \quad a_7 = 7a_1 + 77/60, \\ a_8 = -a_1 + 1/6. \end{cases}$$

Then, equation (10) is transformed into the following equation:

$$b_7\tau^7\phi_k^{(7)} + O(\tau^8) = O(\tau^7). \tag{14}$$

Evidently, to make equation (14) hold true, b_7 should be unequal to 0.

In order to check whether the general 8-instant ZTD formula (2) is zero-stable, with the above solution, we write out its characteristic equation as

$$\begin{aligned} \rho(\lambda) = &(\lambda - 1)\Bigg[-\left(a_1 - \frac{1}{6}\right)\lambda^6 + \left(6a_1 + \frac{29}{20}\right)\lambda^5 \\ & - \left(15a_1 + \frac{21}{20}\right)\lambda^4 + \left(20a_1 + \frac{37}{60}\right)\lambda^3 - \Bigg(15a_1 \\ & + \frac{13}{60}\Bigg)\lambda^2 + \left(6a_1 + \frac{1}{30}\right)\lambda - a_1\Bigg] = 0. \end{aligned} \tag{15}$$

Let

$$\begin{aligned} \rho'(\lambda) = &-\left(a_1 - \frac{1}{6}\right)\lambda^6 + \left(6a_1 + \frac{29}{20}\right)\lambda^5 - \Bigg(15a_1 \\ & + \frac{21}{20}\Bigg)\lambda^4 + \left(20a_1 + \frac{37}{60}\right)\lambda^3 - \left(15a_1 + \frac{13}{60}\right)\lambda^2 \\ & + \left(6a_1 + \frac{1}{30}\right)\lambda - a_1\Bigg] = 0 \end{aligned} \tag{16}$$

Evidently, $\lambda = 1$ lying on the unit circle is one root of equation (15), and other roots of it are determined by equation (16). Hence, if and only if all the roots of (16) lie inside of the unit circle, the general 8-instant ZTD formula (2) with consistency of order 6 is zero-stable. In order to use Routh stability criterion to investigate the roots of equation (16), we adopt Tustin transform (also termed bilinear transform) $\lambda = (1 + \eta)/(1 - \eta)$ to equation (16) [32]. Then, we obtain the following equation:

$$c_6\eta^6 + c_5\eta^5 + c_4\eta^4 + c_3\eta^3 + c_2\eta^2 + c_1\eta^1 + c_0 = 0, \tag{17}$$

where $c_6 = 960a_1 + 48$, $c_5 = 95$, $c_4 = 27$, $c_3 = -100$, $c_2 = -140$, $c_1 = -75$, and $c_0 = -15$.

According to Routh stability criterion [33], if and only if a characteristic equation only has positive coefficients and the numeric values in the first column of the corresponding Routh table are larger than zero, the real parts of all the roots of the equation are negative. To make the general 8-instant ZTD formula (2) have zero-stability, we need to keep all the coefficients of (17) larger than zero and the signs in the first column of the corresponding Routh table unchanged. Table II shows the Routh table for equation (17). Evidently, there

TABLE II
ROUTH TABLE FOR EQUATION (17)

η^6	$960a_1+48$	27	-140	-15
η^5	95	-100	-75	0
η^4	$(19200a_1+1473)/19$	$(14400a_1-1940)/19$	-15	0
η^3	$-(3288000a_1-37000)/(19200a_1+1473)$	$-(480000a_1+27800)/(6400a_1+491)$	0	0
η^2	$(5184000a_1^2-2798400a_1-13439)/(16440a_1-185)$	-15	0	0
η^1	$-(388800000a_1^2-174960000a_1+691200)/(5184000a_1^2$ $-2798400a_1-13439)$	0	0	0
η^0	-15	0	0	0

are some negative coefficients in (17) which violates Routh stability criterion. Thus, there is no appropriate coefficient a_1 to form a zero-stable 8-instant ZTD formula which is consistent of order 6 as well. Thereby, there is no convergent 8-instant ZTD formula with a truncation error being $O(\tau^6)$. Here, we complete the proof. □

Based on Theorems 1 and 2, we conclude that any 8-instant ZTD formula cannot possess sextic precision or higher. Instinctively, we may assume that the general 8-instant ZTD formula (2) converges with order 5. However, we also cannot find the existence of any 8-instant ZTD formula which converges with a fifth order truncation error. Here are the theorem and its proof.

Theorem 3: Assume $\tau\in(0,1)$. Let $O(\tau^4)$ and $O(\tau^5)$ correspond to the 4-th order truncation error and the 5-th order truncation error, respectively. Then, there is no 8-instant ZTD formula converging with order 5 or higher. However, the precision of any convergent 8-instant ZTD formula is quartic or lower.

Proof: Similarly, provided that the general 8-instant ZTD formula (2) is consistent of order 5, then $p=5$ in formula (2) with the following to be satisfied:

$$\begin{cases} b_0=\displaystyle\sum_{i=1}^{8}a_i=0,\\ b_1=-6a_1-5a_2-4a_3-3a_4-2a_5-a_6+a_8-1\\ \quad=0,\\ b_2=18a_1+25a_2/2+8a_3+9a_4/2+2a_5+a_6/2\\ \quad+a_8/2=0,\\ b_3=-36a_1-125a_2/6-32a_3/3-9a_4/2-4a_5/3\\ \quad-a_6/6+a_8/6=0,\\ b_4=54a_1+625a_2/24+32a_3/3+27a_4/8+2a_5/3\\ \quad+a_6/24+a_8/24=0,\\ b_5=-324a_1/5-625a_2/24-128a_3/15-81a_4/40\\ \quad-4a_5/15-a_6/120+a_8/120=0. \end{cases} \tag{18}$$

The solution of condition group (18) is

$$\begin{cases} a_3=-21a_1-6a_2+1/20,\quad a_4=70a_1+15a_2-1/3,\\ a_5=-105a_1-20a_2+1,\quad a_6=84a_1+15a_2-2,\\ a_7=-35a_1-6a_2+13/12,\quad a_8=6a_1+a_2+1/5. \end{cases}$$

With condition group (18) satisfied, equation (10) is transformed into the equation below:

$$b_6\tau^6\phi_k^{(6)}+b_7\tau^7\phi_k^{(7)}+O(\tau^8)=O(\tau^6),$$

where $b_6\neq0$. Furthermore, to explore the zero-stability of formula (2), with the above solution, we write out the characteristic equation of (2) directly:

$$\begin{aligned}\rho(\lambda)=&(\lambda-1)\Bigg[\left(6a_1+a_2+\frac{1}{5}\right)\lambda^6-\bigg(29a_1+5a_2\\&-\frac{77}{60}\bigg)\lambda^5+\left(55a_1+10a_2-\frac{43}{60}\right)\lambda^4-\bigg(50a_1\\&+10a_2-\frac{17}{60}\bigg)\lambda^3+\left(20a_1+5a_2-\frac{1}{20}\right)\lambda^2\\&-(a_1+a_2)\lambda-a_1\Bigg]=0.\end{aligned} \tag{19}$$

Let

$$\begin{aligned}\rho'(\lambda)=&\left(6a_1+a_2+\frac{1}{5}\right)\lambda^6-\left(29a_1+5a_2-\frac{77}{60}\right)\lambda^5\\&+\left(55a_1+10a_2-\frac{43}{60}\right)\lambda^4-\bigg(50a_1+10a_2\\&-\frac{17}{60}\bigg)\lambda^3+\left(20a_1+5a_2-\frac{1}{20}\right)\lambda^2-(a_1\\&+a_2)\lambda-a_1=0.\end{aligned} \tag{20}$$

After adopting Tustin transform $\lambda=(1+\eta)/(1-\eta)$ to equation (20), we obtain the equation as follows:

$$d_6\eta^6+d_5\eta^5+d_4\eta^4+d_3\eta^3+d_2\eta^2+d_1\eta+d_0=0, \tag{21}$$

where $d_6=2400a_1+480a_2-32$, $d_5=3360a_1+480a_2-79$, $d_4=-27$, $d_3=100$, $d_2=140$, $d_1=75$ and $d_0=15$. Then, we calculate out the Routh table for equation (21), which is represented in Table III. In the light of Routh stability criterion [33], we determine that there are no appropriate coefficients

TABLE III
ROUTH TABLE FOR EQUATION (21)

η^6	$2400a_1+480a_2-32$	-27	140	15
η^5	$3360a_1+480a_2-79$	100	75	0
η^4	$-(330720a_1+60960a_2-5333)/(3360a_1+480a_2-79)$	$(290400a_1+31200a_2-8660)/(3360a_1+480a_2-79)$	15	0
η^3	$(975744000a_1^2+244224000a_1a_2-18967200a_1+14976000a_2^2-525600a_2+150840)/(330720a_1+60960a_2-5333)$	$(169344000a_1^2+48384000a_1a_2+16840800a_1+3456000a_2^2+3434400a_2-306360)/(330720a_1+60960a_2-5333)$	0	0
η^2	$(841670400a_1^2+196454400a_1a_2-3255220a_1+11769600a_2^2+2715540a_2-34551)/(8131200a_1^2+2035200a_1a_2-158060a_1+124800a_2^2-4380a_2+1257)$	15	0	0
η^1	$(71124480000a_1^3+30481920000a_1^2a_2+56329344000a_1^2+4354560000a_1a_2^2+13656384000a_1a_2-302025600a_1+207360000a_2^3+867456000a_2^2+157939200a_2-1451520)/(841670400a_1^2+196454400a_1a_2-3255220a_1+11769600a_2^2+2715540a_2-34551)$	0	0	0
η^0	15	0	0	0

TABLE IV
JURY TABLE FOR EQUATION (20)

j_{11}	j_{12}	j_{13}	j_{14}	j_{15}	j_{16}	j_{17}
j_{17}	j_{16}	j_{15}	j_{14}	j_{13}	j_{12}	j_{11}
j_{21}	j_{22}	j_{23}	j_{24}	j_{25}	j_{26}	
j_{26}	j_{25}	j_{24}	j_{23}	j_{22}	j_{21}	
j_{31}	j_{32}	j_{33}	j_{34}	j_{35}		
j_{35}	j_{34}	j_{33}	j_{32}	j_{31}		
j_{41}	j_{42}	j_{43}	j_{44}			
j_{44}	j_{43}	j_{42}	j_{41}			
j_{51}	j_{52}	j_{53}				

a_1 and a_2 to guarantee that the general 8-instant ZTD formula (2) with consistency of order 5 is zero-stable. In other words, there is no 8-instant ZTD formula converging with order 5.

We can also use Jury stability criterion [34] to prove this theorem. We build the Jury table for equation (19), and its modality is shown in Table IV. Because it is too complicated, we cannot list all items. However, we list some items related to the zero-stable conditions of the general 8-instant ZTD formula (2) as follows:

$$
\begin{aligned}
j_{11} =& -a_1,\\
j_{17} =& \frac{1}{5}(30a_1+5a_2+1),\\
j_{21} =& \frac{1}{25}(35a_1+5a_2+1)(25a_1+5a_2+1),\\
j_{26} =& -\frac{1}{60}(2100a_1^2+720a_1a_2-65a_1+60a_2^2+12a_2),\\
j_{31} =& \frac{1}{90000}(1045a_1+60a_2+12)(21000a_1^2+7200a_1a_2\\
&+395a_1+600a_2^2+180a_2+12),\\
j_{35} =& \frac{1}{90000}(70245000a_1^3+31119000a_1^2a_2\\
&+1866150a_1^2+4419000a_1a_2^2+801575a_1a_2\\
&-16445a_1+201000a_2^3+79500a_2^2+7680a_2-36),\\
j_{41} =& -\frac{1}{64800000}(386400a_1^2+101400a_1a_2-5845a_1\\
&+6600a_2^2+1140a_2-36)(2634000a_1^2+763800a_1\\
&a_2-2945a_1+47400a_2^2+10020a_2+108),\\
j_{44} =& \frac{1}{21600000}(20638800000a_1^4+540960000a_1^3a_2
\end{aligned}
$$

$$
\begin{aligned}
&-9546354000a_1^3-2780760000a_1^2a_2^2\\
&-3724028000a_1^2a_2-147624075a_1^2-574080000a_1\\
&a_2^3-480550000a_1a_2^2-67527400a_1a_2+1123880a_1\\
&-32280000a_2^4-26856000a_2^3-6764400a_2^2\\
&-516000a_2+4176),
\end{aligned}
$$

$$
\begin{aligned}
j_{51}=&\frac{1}{1166400000}(22758600a_1^2+5544600a_1a_2\\
&-139855a_1+341400a_2^2+64860a_2-684)\\
&(51660000a_1^3+24012000a_1^2a_2-2754600a_1^2\\
&+3636000a_1a_2^2-551100a_1a_2+8725a_1\\
&+180000a_2^3-3000a_2^2-7620a_2+36),
\end{aligned}
$$

and

$$
\begin{aligned}
j_{53}=&\frac{1}{1458000000}(1542817017000000a_1^5\\
&+1093725617400000a_1^4a_2-29518892910000a_1^4\\
&+306796825800000a_1^3a_2^2-3677841585000a_1^3a_2\\
&+313041197250a_1^3+42644982600000a_1^2a_2^3\\
&+2355038100000a_1^2a_2^2-25376899500a_1^2a_2\\
&+3084390025a_1^2+2941693200000a_1a_2^4\\
&+535822740000a_1a_2^3-2185494000a_1a_2^2\\
&+1539411000a_1a_2-24825240a_1+80654400000\\
&a_2^5+30186360000a_2^4+2679408000a_2^3+28040400\\
&a_2^2+10415520a_2-92016).
\end{aligned}
$$

According to Jury stability criterion [34], we list the following condition group which is sufficient and necessary for the general 8-instant ZTD formula (2) to have zero-stability and consistency of order 5 as

$$
\begin{cases}
\rho'(1)>0,\\
(-1)^6\rho'(-1)>0,\\
|j_{11}|<j_{17},\\
|j_{21}|>|j_{26}|,\\
|j_{31}|>|j_{35}|,\\
|j_{41}|>|j_{44}|,\\
|j_{51}|>|j_{53}|.
\end{cases}
\tag{22}
$$

After calculation, $\rho'(1)=1>0$. However, there is no solution that satisfies the second condition through the fifth one of condition group (22). The solving process of (22) is displayed in Fig. 1. In each subfigure, the red region represents the feasible solution region for condition(s) in condition group (22). As seen from Fig. 1(d), there is no red region, which means that there are no proper coefficients a_1 and a_2 for the general 8-instant ZTD formula (2) converging with order 5 as well. On the basis of Theorems 1 and 2, there is no convergent 8-instant ZTD formula with a truncation error being $O(\tau^5)$ or smaller. Moreover, in the manner of construction-type proof, according to [30] as well as Table I, there is at least one 8-instant ZTD formula converging with a truncation error being $O(\tau^4)$. Hence, any 8-instant ZTD formula converges with a truncation error being $O(\tau^4)$ or greater. Here, the proof is thus completed. □

IV. Conclusion

As pointed out in this paper, ZTD formulas have been very useful in the time discretization of continuous-time models. Besides, the precision of a ZTD formula influences the error of the corresponding discrete model. Thus, it has been instinctive for us to seek more precise ZTD formulas. Moreover, we have discovered that any convergent 8-instant ZTD formula cannot possess quintic precision or higher, with the proof in this paper. In conclusion, the truncation error with which any 8-instant ZTD formula converges is $O(\tau^4)$ or greater.

Appendix

There are three results on the convergent conditions of the discrete linear m-step method [35], [36], with $g=\tau$ correspondingly in this paper. They are listed as below.

Result 1: A discrete-time linear m-step method has consistency (i.e., is consistent) of order p if its truncation error is $O(g^{p+1})$ with $p>0$.

Result 2: For a discrete-time m-step method $\sum_{i=0}^{m}\alpha_i\phi_{k+i}=g\sum_{i=0}^{m}\beta_i\psi_{k+i}$, if all the roots ι of its characteristic equation $P_m(\iota)=\sum_{i=0}^{m}\alpha_i\iota^i$ satisfy $|\iota|\leq 1$ with $|\iota|=1$ being simple (precisely, unique), then the m-step method has zero-stability (i.e., is zero-stable).

Result 3: A discrete-time linear m-step method is convergent with the order of its truncation error, i.e., $x_{[t/g]}\to x^*(t)$, for all $t\in[0,t_{\mathrm{f}}]$, as $g\to 0$, if and only if the method is consistent and zero-stable. In other words, consistency plus zero-stability means convergence.

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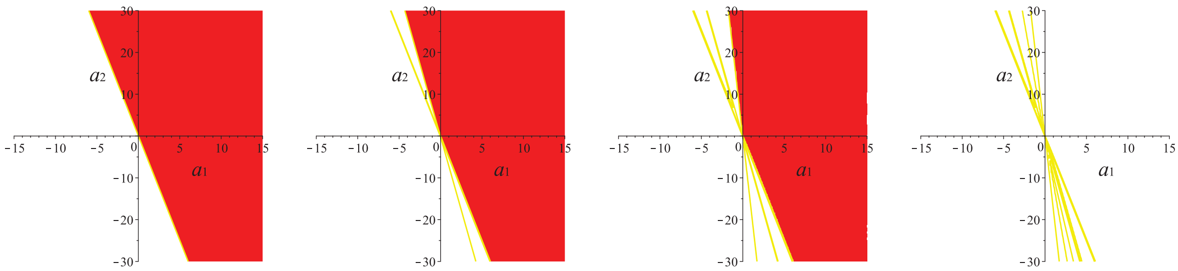

(a) (b) (c) (d)

Fig. 1. Solving process of condition group (22), where red regions are feasible solution regions. (a) Satisfaction for condition 2. (b) Satisfaction for conditions 2 and 3. (c) Satisfaction for conditions 2 through 4. (d) No satisfaction for conditions 2 through 5.

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